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## ON A SEQUENCE INVOLVING SUMS OF PRIMES

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**ABSTRACT.** For  $n = 1, 2, 3, \dots$  let  $S_n$  be the sum of the first  $n$  primes. We mainly show that the sequence  $a_n = \sqrt[n]{S_n/n}$  ( $n = 1, 2, 3, \dots$ ) is strictly decreasing, and moreover the sequence  $a_{n+1}/a_n$  ( $n = 10, 11, \dots$ ) is strictly increasing. We also formulate similar conjectures involving twin primes or partitions of integers.

### 1. INTRODUCTION

The mysterious prime numbers are very important in number theory due to the Fundamental Theorem of Arithmetic (cf. [IR]). For problems and results on primes, the reader may consult the excellent book [CP] by R. Crandall and C. Pomerance.

For  $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$  let  $p_n$  denote the  $n$ th prime. The unsolved Firoozbakht conjecture (cf. [R, p. 185]) asserts that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+,$$

i.e., the sequence  $(\sqrt[n]{p_n})_{n \geq 1}$  is strictly decreasing. This implies the inequality  $p_{n+1} - p_n < \log^2 p_n - \log p_n$  for large  $n$ , which is even stronger than Cramer's conjecture. Let  $P_n$  be the product of the first  $n$  primes. Then  $P_n < p_{n+1}^n$  and hence  $P_{n+1}^{n+1} < P_n^{n+1}$ . So the sequence  $(\sqrt[n]{P_n})_{n \geq 1}$  is strictly increasing.

Now let us look at a simple example not related to primes.

*Example 1.1.* Let  $a_n = \sqrt[n]{n}$  for  $n \in \mathbb{Z}^+$ . Then the sequence  $(a_n)_{n \geq 3}$  is strictly decreasing, and the sequence  $(a_{n+1}/a_n)_{n \geq 4}$  is strictly increasing. To

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see this we investigate the function  $f(x) = \log(x^{1/x}) = (\log x)/x$  with  $x \geq 3$ . As  $f'(x) = (1 - \log x)/x^2 < 0$ , we have  $f(n) > f(n+1)$  for  $n = 3, 4, \dots$ . Since

$$f''(x) = \frac{2 \log x - 3}{x^3} > 0 \quad \text{for } x \geq 4.5,$$

the function  $f(x)$  is strictly convex over the interval  $(4.5, +\infty)$  and so

$$2f(n+1) < f(n) + f(n+2) \quad (\text{i.e., } a_{n+1}^2 < a_n a_{n+2}) \quad \text{for } n = 5, 6, \dots$$

The inequality  $a_5^2 < a_4 a_6$  can be verified directly.

A sequence  $(a_n)_{n \geq 1}$  of nonnegative real numbers is said to be *log-convex* if  $a_{n+1}^2 \leq a_n a_{n+2}$  for all  $n = 1, 2, 3, \dots$ . Many combinatorial sequences (such as the sequence of the Catalan numbers) are log-convex, the reader may consult [LW] for some results on log-convex sequences.

For  $n \in \mathbb{Z}^+$  let  $S_n = \sum_{k=1}^n p_k$  be the sum of the first  $n$  primes. For instance,

$$S_1 = 2, \quad S_2 = 2 + 3 = 5, \quad S_3 = 2 + 3 + 5 = 10, \quad S_4 = 2 + 3 + 5 + 7 = 17.$$

Recently the author [S] conjectured that for any positive integer  $n$  the interval  $(S_n, S_{n+1})$  contains a prime. As  $S_n < np_{n+1}$  for all  $n \in \mathbb{Z}^+$ , the sequence  $(S_n/n)_{n \geq 1}$  is strictly increasing.

In this paper we mainly establish the following result.

**Theorem 1.1.** *The sequences  $(\sqrt[n]{S_n})_{n \geq 2}$  and  $(\sqrt[n]{S_n/n})_{n \geq 1}$  are strictly decreasing.*

*Remark 1.1.* Note that  $S_n/n$  is just the arithmetic mean of the first  $n$  primes. It is interesting to compare Theorem 1.1 with Firoozbakht's conjecture that  $(\sqrt[n]{p_n})_{n \geq 1}$  is strictly decreasing.

For  $\alpha > 0$  and  $n \in \mathbb{Z}^+$  define

$$S_n^{(\alpha)} = \sum_{k=1}^n p_k^\alpha.$$

We actually obtain the following extension of Theorem 1.1.

**Theorem 1.2.** *Let  $\alpha \geq 1$  and  $n \in \mathbb{Z}^+$  with  $n \geq \max\{100, e^{2 \times 1.348^\alpha + 1}\}$ . Then*

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(\alpha)}}{n+1}} \quad (1.1)$$

and hence

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}. \quad (1.2)$$

*Remark 1.2.* In view of Example 1.1, (1.1) implies (1.2) if  $n \geq 3$ . We conjecture that (1.1) holds for any  $\alpha > 0$  and  $n \in \mathbb{Z}^+$ .

Note that  $\lfloor e^{2 \times 1.348 + 1} \rfloor = 40$  and we can easily verify that

$$\sqrt[n]{\frac{S_n}{n}} > \sqrt[n+1]{\frac{S_{n+1}}{n+1}} \quad \text{for every } n = 1, \dots, 99.$$

So Theorem 1.1 follows from Theorem 1.2 in the case  $\alpha = 1$ .

**Corollary 1.1.** *For each  $\alpha \in \{2, 3, 4\}$ , the sequences*

$$\left( \sqrt[n]{\frac{S_n^{(\alpha)}}{n}} \right)_{n \geq 1} \quad \text{and} \quad \left( \sqrt[n]{S_n^{(\alpha)}} \right)_{n \geq 1}$$

*are strictly decreasing.*

*Proof.* Observe that

$$\lfloor e^{2 \times 1.348^2 + 1} \rfloor = 102, \quad \lfloor e^{2 \times 1.348^3 + 1} \rfloor = 364, \quad \lfloor e^{2 \times 1.348^4 + 1} \rfloor = 2005.$$

In light of Theorem 1.2 and Example 1.1, it suffices to verify that

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(2)}}{n+1}}$$

whenever  $\alpha \in \{2, 3, 4\}$  and  $n \in \{1, \dots, \lfloor e^{2 \times 1.348^\alpha + 1} \rfloor\}$ . These can be easily done via computer.  $\square$

Our following theorem is more sophisticated than Theorem 1.2.

**Theorem 1.3.** *Let  $\alpha \geq 1$ . Then the sequence*

$$\left( \sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)} / \sqrt[n]{S_n^{(\alpha)}/n} \right)_{n \geq N(\alpha)}$$

*is strictly increasing, where*

$$N(\alpha) = \max \left\{ 350000, \lceil e^{((\alpha+1)^2 1.2^{2\alpha+1} + (\alpha+1) 1.2^{\alpha+1} + 1)/\alpha} \rceil \right\}. \quad (1.3)$$

**Corollary 1.2.** *The sequences*

$$\begin{aligned} & \left( \sqrt[n+1]{S_{n+1}/(n+1)} / \sqrt[n]{S_n/n} \right)_{n \geq 10}, \quad \left( \sqrt[n+1]{S_{n+1}} / \sqrt[n]{S_n} \right)_{n \geq 5}, \\ & \left( \sqrt[n+1]{S_{n+1}^{(2)}/(n+1)} / \sqrt[n]{S_n^{(2)}/n} \right)_{n \geq 13}, \quad \left( \sqrt[n+1]{S_{n+1}^{(2)}} / \sqrt[n]{S_n^{(2)}} \right)_{n \geq 10}, \\ & \left( \sqrt[n+1]{S_{n+1}^{(3)}/(n+1)} / \sqrt[n]{S_n^{(3)}/n} \right)_{n \geq 17}, \quad \left( \sqrt[n+1]{S_{n+1}^{(3)}} / \sqrt[n]{S_n^{(3)}} \right)_{n \geq 10}, \\ & \left( \sqrt[n+1]{S_{n+1}^{(4)}/(n+1)} / \sqrt[n]{S_n^{(4)}/n} \right)_{n \geq 35}, \quad \left( \sqrt[n+1]{S_{n+1}^{(4)}} / \sqrt[n]{S_n^{(4)}} \right)_{n \geq 17} \end{aligned}$$

*are all strictly increasing.*

*Proof.* For  $N(\alpha)$  given by (1.3), via computation we find that

$$N(1) = 350000, \quad N(2) = 1606294, \quad N(3) = 4415694368$$

and

$$N(4) = 2916110783693881.$$

Via computer we can verify that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)/(n+1)}}}{\sqrt[n]{S_n^{(\alpha)}/n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)/(n+2)}}}{\sqrt[n+1]{S_{n+1}^{(\alpha)/(n+1)}}}$$

for all  $\alpha \in \{1, 2, 3, 4\}$  and  $n = N_0(\alpha), \dots, N(\alpha) - 1$ , where

$$N_0(1) = 10, \quad N_0(2) = 13, \quad N_0(3) = 17, \quad N_0(4) = 35.$$

Combining this with Theorem 1.3 we obtain that

$$\left( \sqrt[n+1]{S_{n+1}^{(\alpha)/(n+1)}} / \sqrt[n]{S_n/n} \right)_{n \geq N_0(\alpha)}$$

is strictly increasing for each  $\alpha = 1, 2, 3, 4$ . Recall that  $(\sqrt[n+1]{n+1} / \sqrt[n]{n})_{n \geq 4}$  is strictly increasing by Example 1.1. So  $(\sqrt[n+1]{S_{n+1}^{(\alpha)/(n+1)}} / \sqrt[n]{S_n/n})_{n \geq N_0(\alpha)}$  is strictly increasing for any  $\alpha \in \{1, 2, 3, 4\}$ . It remains to check that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}{\sqrt[n]{S_n^{(\alpha)}}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}$$

for all  $\alpha \in \{1, 2, 3, 4\}$  and  $n = n_0(\alpha), \dots, N_0(\alpha) - 1$ , where  $n_0(1) = 5$ ,  $n_0(2) = 10$ ,  $n_0(3) = 10$ , and  $n_0(4) = 17$ . This can be easily done via computer.  $\square$

We conclude this section by posing three new conjectures.

**Conjecture 1.1.** *The two constants*

$$s_1 = \sum_{n=1}^{\infty} \frac{1}{S_n} \quad \text{and} \quad s_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{S_n}$$

*are both transcendental numbers.*

*Remark 1.3.* Our computation shows that  $s_1 \approx 1.023476$  and  $s_2 \approx -0.3624545778$ .

If  $p$  and  $p+2$  are both primes, then they are called twin primes. The famous twin prime conjecture states that there are infinitely many twin primes.

**Conjecture 1.2.** (i) *If  $\{t_1, t_1 + 2\}, \dots, \{t_n, t_n + 2\}$  are the first  $n$  pairs of twin primes, then the first prime  $t_{n+1}$  in the next pair of twin primes is smaller than  $t_n^{1+1/n}$ , i.e.,  $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$ .*

(ii) *The sequence  $(\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n})_{n \geq 9}$  is strictly increasing with limit 1, where  $T_n = \sum_{k=1}^n t_k$ .*

*Remark 1.4.* Via **Mathematica** the author has verified that  $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$  for all  $n = 1, \dots, 500000$ , and  $\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n} < \sqrt[n+2]{T_{n+2}}/\sqrt[n+1]{T_{n+1}}$  for all  $n = 9, 10, \dots, 500000$ . Note that  $t_{500000} = 115438667$ .

Recall that a partition of a positive integer  $n$  is a way of writing  $n$  as a sum of positive integers with the order of addends ignored. Also, a *strict partition* of  $n \in \mathbb{Z}^+$  is a way of writing  $n$  as a sum of *distinct* positive integers with the order of addends ignored. For  $n = 1, 2, 3, \dots$  we denote by  $p(n)$  and  $p_*(n)$  the number of partitions of  $n$  and the number of strict partitions of  $n$  respectively. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad p_*(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as } n \rightarrow +\infty$$

(cf. [HR] and [AS, p. 826]) and hence  $\lim_{n \rightarrow \infty} \sqrt[n]{p(n)} = \lim_{n \rightarrow \infty} \sqrt[n]{p_*(n)} = 1$ . Here we formulate a conjecture similar to Conjecture 1.1.

**Conjecture 1.3.** *For  $n \in \mathbb{Z}^+$  let*

$$q(n) = \frac{p(n)}{n}, \quad q_*(n) = \frac{p_*(n)}{n}, \quad r(n) = \sqrt[n]{q(n)}, \quad \text{and} \quad r_*(n) = \sqrt[n]{q_*(n)}.$$

*Then the sequences  $(q(n+1)/q(n))_{n \geq 31}$  and  $(q_*(n+1)/q_*(n))_{n \geq 44}$  are strictly decreasing, and the sequences  $(r(n+1)/r(n))_{n \geq 60}$  and  $(r_*(n+1)/r_*(n))_{n \geq 120}$  are strictly increasing.*

*Remark 1.5.* Via **Mathematica** we have verified the conjecture for  $n$  up to  $10^5$ . In light of Example 1.1, Conjecture 1.3 implies that the sequences

$$\left(\frac{p(n+1)}{p(n)}\right)_{n \geq 25}, \quad \left(\frac{p_*(n+1)}{p_*(n)}\right)_{n \geq 32}, \quad (\sqrt[n]{p(n)})_{n \geq 6}, \quad (\sqrt[n]{p_*(n)})_{n \geq 9}$$

are all strictly decreasing, and that the sequences  $(\sqrt[n+1]{p(n+1)}/\sqrt[n]{p(n)})_{n \geq 26}$  and  $(\sqrt[n+1]{p_*(n+1)}/\sqrt[n]{p_*(n)})_{n \geq 45}$  are strictly increasing. The fact that  $(p(n+1)/p(n))_{n \geq 25}$  is strictly decreasing was conjectured by W.Y.C. Chen [C] and proved by J.E. Janoski [J, pp. 7-23].

## 2. PROOFS OF THEOREMS 1.2 AND 1.3

**Lemma 2.1.** *Let  $\alpha \geq 1$  and  $n \in \{2, 3, \dots\}$ . Then*

$$S_n^{(\alpha)} > 2^\alpha + \frac{n^{\alpha+1} \log^\alpha n}{\alpha + 1} \left( 1 - \frac{\alpha}{(\alpha + 1) \log n} \right). \quad (2.1)$$

*Proof.* It is known that  $p_k \geq k \log k$  for  $k = 2, 3, \dots$  (cf. [R] and [RS, (3.12)]). Thus

$$S_n^{(\alpha)} - 2^\alpha = \sum_{k=2}^n p_k \geq \sum_{k=2}^n (k \log k)^\alpha > \sum_{k=2}^n \int_{k-1}^k (x \log x)^\alpha dx = \int_1^n (x \log x)^\alpha dx$$

Using integration by parts, we find that

$$\begin{aligned} \int_1^n (x \log x)^\alpha dx &= \frac{x^{\alpha+1}}{\alpha+1} \log^\alpha x \Big|_{x=1}^n - \int_1^n \left( \frac{x^{\alpha+1}}{\alpha+1} \cdot \frac{\alpha(\log x)^{\alpha-1}}{x} \right) dx \\ &= \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha}{\alpha+1} \int_1^n x^\alpha (\log x)^{\alpha-1} dx \\ &\geq \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha}{\alpha+1} \int_1^n x^\alpha (\log n)^{\alpha-1} dx \\ &= \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha n^{\alpha+1}}{(\alpha+1)^2} (\log n)^{\alpha-1}. \end{aligned}$$

Therefore (2.1) holds.  $\square$

**Lemma 2.2.** *Let  $\alpha \geq 1$  and  $n \in \mathbb{Z}^+$  with  $n \geq 55$ . Then*

$$\log S_n^{(\alpha)} > (\alpha + 1) \log n. \quad (2.2)$$

*Proof.* Note that  $54 < e^4 < 55 \leq n$ . As  $\log^\alpha n > 4^\alpha = (2^\alpha)^2 \geq (\alpha + 1)^2$ , by Lemma 2.1 we have

$$S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^\alpha n}{\alpha + 1} \left( 1 - \frac{\alpha}{\alpha + 1} \right) = \frac{n^{\alpha+1}}{(\alpha + 1)^2} \log^\alpha n \geq n^{\alpha+1}$$

and hence (2.2) follows.  $\square$

*Proof of Theorem 1.2.* It is known that

$$p_m < m(\log m + \log \log m)$$

for any  $m \geq 6$  (cf. [RS, (3.13)] and [D, Lemma 1]). If  $m \geq 101$ , then

$$\frac{\log \log m}{\log m} \leq \frac{\log \log 101}{\log 101} < 0.3314$$

and hence  $p_m < 1.3314m \log m$ . As  $n+1 \leq 1.01n$ , we have

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log((n+1)/n)}{\log n} \leq 1 + \frac{\log 1.01}{\log n} \leq 1 + \frac{\log 1.01}{\log 100} < 1.0022.$$

Therefore

$$p_{n+1} < 1.3314(n+1) \log(n+1) < 1.3314 \times 1.01n \times 1.0022 \log n < 1.348n \log n.$$

Combining Lemmas 2.1 and 2.2, we see that

$$\begin{aligned} & S_n^{(\alpha)} \left( \frac{n+1}{n^{1+1/n}} \sqrt[n]{S_n^{(\alpha)}} - 1 \right) \\ &= S_n^{(\alpha)} \left( e^{(\log S_n^{(\alpha)})/n + \log(n+1) - (1+1/n) \log n} - 1 \right) \\ &\geq S_n^{(\alpha)} \left( e^{(\log S_n^{(\alpha)} - \log n)/n} - 1 \right) \geq S_n^{(\alpha)} \left( e^{(\alpha \log n)/n} - 1 \right) \\ &> \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} \left( 1 - \frac{\alpha}{(\alpha+1) \log n} \right) \frac{\alpha \log n}{n} \\ &= \frac{\alpha}{\alpha+1} (n \log n)^\alpha \left( \log n - \frac{\alpha}{\alpha+1} \right) \\ &> \frac{(n \log n)^\alpha}{2} (\log n - 1). \end{aligned}$$

As  $(\log n - 1)/2 \geq 1.348^\alpha$ , from the above we get

$$(n+1) \left( \frac{S_n^{(\alpha)}}{n} \right)^{1+1/n} - S_n^{(\alpha)} > (1.348n \log n)^\alpha > p_{n+1}^\alpha$$

and hence

$$\left( \frac{S_n^{(\alpha)}}{n} \right)^{(n+1)/n} > \frac{S_{n+1}^{(\alpha)}}{n+1}$$

which yields (1.1). As mentioned in Remark 1.2, (1.2) follows from (1.1). This concludes the proof.  $\square$

*Proof of Theorem 1.3.* Fix an integer  $n \geq N(\alpha)$ . For any integer  $m \geq 350001$ , we have

$$\frac{\log \log m}{\log m} \leq \frac{\log \log 350001}{\log 350001} < 0.1996$$

and hence

$$p_m < m(\log m) \left( 1 + \frac{\log \log m}{\log m} \right) < 1.1996m \log m.$$

As  $n \geq 350000$ , we have

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log(1+1/n)}{\log n} \leq \frac{\log 350001}{\log 350000} < 1 + 10^{-6}.$$

Therefore

$$\begin{aligned} p_{n+1} &< 1.1996(n+1) \log(n+1) \\ &< 1.1996 \times \frac{350001}{350000} n \times (1 + 10^{-6}) \log n < 1.2n \log n. \end{aligned}$$

Since  $\log n \geq \log 350000 > 1/0.078335$ , Lemma 2.1 implies that

$$S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} (1 - 0.078335) > \frac{n^{\alpha+1} \log^\alpha n}{1.085(\alpha+1)}.$$

Therefore

$$q_n^{(\alpha)} := \frac{p_{n+1}^\alpha}{S_n^{(\alpha)}} < \frac{c_\alpha}{n}, \quad (2.3)$$

where  $c_\alpha = 1.085(\alpha+1)1.2^\alpha$ .

By calculus,

$$x - \frac{x^2}{2} < \log(1+x) < x \quad \text{for } x > 0$$

and

$$-x - x^2 < \log(1-x) < -x \quad \text{for } 0 < x < 0.5.$$

Thus

$$\log \frac{S_{n+1}^{(\alpha)}/(n+1)}{S_n^{(\alpha)}/n} = \log \left( 1 - \frac{1}{n+1} \right) + \log(1 + q_n^{(\alpha)}) < -\frac{1}{n+1} + q_n^{(\alpha)}$$

and

$$\begin{aligned} \log \frac{S_{n+2}^{(\alpha)}/(n+2)}{S_n^{(\alpha)}/n} &= \log \left( 1 - \frac{2}{n+2} \right) + \log(1 + 2q_n^{(\alpha)}) \\ &> -\frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2. \end{aligned}$$

Hence

$$\begin{aligned} D_n^{(\alpha)} &:= \frac{2}{n+1} \log \frac{S_{n+1}^{(\alpha)}}{n+1} - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n+2} \log \frac{S_{n+2}^{(\alpha)}}{n+2} \\ &< \frac{2}{n+1} \left( \log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n+1} + q_n^{(\alpha)} \right) - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n} \\ &\quad - \frac{1}{n+2} \left( \log \frac{S_n^{(\alpha)}}{n} - \frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2 \right) \\ &= \frac{-2 \log(S_n^{(\alpha)}/n)}{n(n+1)(n+2)} - \frac{2}{(n+1)^2} + \frac{2}{(n+2)^2} + \frac{4}{(n+2)^3} + \frac{2q_n^{(\alpha)}}{(n+1)(n+2)} + \frac{2(q_n^{(\alpha)})^2}{n+2}. \end{aligned}$$



Combining this with (2.2) and (2.3) and noting that  $(350001/350000)n^2 \geq n(n+1)$ , we obtain

$$\begin{aligned}
D_n^{(\alpha)} &< \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{2n+3}{(n+1)^2(n+2)^2} + \frac{4}{(n+2)^3} \\
&\quad + \frac{2c_\alpha}{n(n+1)(n+2)} + \frac{2c_\alpha^2}{n^2(n+2)} \\
&< \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{2}{(n+1)(n+2)^2} + \frac{4}{(n+1)(n+2)^2} \\
&\quad + \frac{2c_\alpha + 2(350001/350000)c_\alpha^2}{n(n+1)(n+2)} \\
&< \frac{-2\alpha \log n}{n(n+1)(n+2)} + \frac{2 + 2c_\alpha + 2(350001/350000)c_\alpha^2}{n(n+1)(n+2)}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{350001}{350000}c_\alpha^2 + c_\alpha + 1 \\
&= \frac{350001}{350000} \times 1.085^2(\alpha+1)^2 1.2^{2\alpha} + 1.085(\alpha+1) 1.2^\alpha + 1 \\
&< 1.2(\alpha+1)^2 1.2^{2\alpha} + 1.2(\alpha+1) 1.2^\alpha + 1 \leq \alpha \log N(\alpha) \leq \alpha \log n.
\end{aligned}$$

So we have  $D_n^{(\alpha)} < 0$  and hence

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}{\sqrt[n]{S_n^{(\alpha)}/n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}/(n+2)}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}$$

as desired.  $\square$

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